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TWO ARBITRARILY SITUATED CRACKS IN AN ELASTIC PLATE UNDER FLEXURE

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Abstract—Based on Reissner's theory for the bending of thin plates and by replacing the cracks by dislocation arrays, the flexure problem for an unbounded plate, containing two arbitrarily situated rectilinear cracks, is reduced to a system of singular integral equations relative to six functions which characterize the density of dislocations.

The solution, in the form of the product of the series of Chebyshev polynomials of the first kind and their weight function, is obtained for the cases of plain bending and uniform twisting along an arbitrarily inclined direction to the cracks.

Numerical results are shown for two fundamental cases of crack configuration, i.e. a pair of equal colinear cracks and equal parallel cracks without stagger.

1. INTRODUCTION

One of the most basic requirements in the fracture mechanics is the knowledge of the singular character of the stress field near the crack tip and many investigations have been made for the crack problems of longitudinal shear, plane strain and plane stress and classical plate bending.

As has already been pointed out [1–4], the classical theory for flexure of plates fails to provide an accurate estimate of the stresses in the neighborhood of the crack, since the classical theory can not satisfy all of the three physically natural boundary conditions along a free edge. This discrepancy, however, can be overcome by using Reissner's theory [5] in which all three boundary conditions on the rim of the plate can be satisfied.

In a previous paper [6], the center of dislocations in a thin plate under flexure has been defined and its expression has been obtained in the framework of Reissner's theory. By replacing the crack by continuous arrays of dislocation, a system of singular integral equations for the dislocation densities can readily be set up. By employing this technique, the plain bending problem of a thin semi-infinite plate weakened by a transverse crack has been discussed [7].

The present paper continues the previous sequence of investigations and deals with the elastic interaction of two arbitrarily situated rectilinear cracks in an infinite plate under plain bending or uniform twisting along an arbitrarily inclined direction to the cracks. By the same approach, the problem is reduced to a set of singular integral equations relative to six unknown functions which characterize the density of dislocations on the crack lines. By the use of the method developed by Erdogan [8], one can obtain the solution of the set of integral equations, in which the essential feature of the singularity of the unknown functions is preserved and the stress intensity factors can easily be estimated. Numerical results are given for two fundamental cases of crack configuration, i.e. a pair of equal colinear cracks and equal parallel cracks without stagger.

2. BASIC EQUATIONS

Consider an isotropic and homogeneous plate of constant thickness h and take its middle plane, before bending occurs, as the x, y plane and denote the thickness coordinate by Z. In the framework of Reissner's theory, the weighted deflection $w(x, y)$ of the plate free from lateral loads should be a plane biharmonic function and can be expressed as

$$Dw(x, y) = [\bar{z}f(z) + z\overline{f(z)} + g(z) + \overline{g(z)}]/2, \tag{1}$$

where $z = x + iy$ and $\bar{z} = x - iy$. Two complex deflection functions $f(z)$ and $g(z)$ are holomorphic in the region occupied by the plate and $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate with Young's modulus E and Poisson's ratio ν. The average rotations β_x and β_y about y- and x-axes respectively are given by

$$D(\beta_x + i\beta_y) = -[f(z) + \overline{zf'(z)} + 2(\kappa+1)\epsilon^2\overline{f''(z)} + \overline{h(z)}] - i(\kappa+1)\epsilon^2\frac{\partial\psi}{\partial\bar{z}}, \tag{2}$$

where $h(z) = g'(z)$, $\kappa = (3+\nu)/(1-\nu)$ and $\epsilon = h/\sqrt{(10)}$. The stress function $\psi(z,\bar{z})$ is the solution of Helmholtz equation

$$\psi - \epsilon^2\nabla^2\psi = 0. \tag{3}$$

The bending moments M_x, M_y, twisting moment H_{xy} and shearing forces V_x, V_y, all per unit length, can be expressed in terms of $f(z)$, $h(z)$ and $\psi(z,\bar{z})$ as follows:

$$\begin{aligned}
M_x + M_y &= -\frac{4(\kappa-1)}{\kappa+1}[f'(z) + \overline{f'(z)}],\\
M_x - M_y - i2H_{xy} &= -\frac{8}{\kappa+1}[h'(z) + \bar{z}f''(z) + 2(\kappa+1)\epsilon^2 f'''(z)] + i8\epsilon^2\frac{\partial^2\psi}{\partial z^2},\\
V_x - iV_y &= -4f''(z) + i2\frac{\partial\psi}{\partial z}.
\end{aligned} \tag{4}$$

Thus the problem is reduced to the construction of the complex deflection functions $f(z)$, $h(z)$ and the stress function $\psi(z,\bar{z})$ which satisfy all of the boundary conditions of the problem.

Moreover the formulas of transformation of stress resultants due to the rotation of coordinate axes $z_1 = e^{-i\beta}z$ are given by

$$\begin{aligned}
&M_{x_1} + M_{y_1} = M_x + M_y,\\
&2(M_{y_1} + iH_{x_1y_1}) = M_x + M_y - e^{i2\beta}(M_x - M_y - i2H_{xy}),\\
&V_{x_1} - iV_{y_1} = e^{i\beta}(V_x - iV_y).
\end{aligned} \tag{5}$$

3. CONTINUOUS DISLOCATION ARRAYS

Consider now the case where three kinds of dislocations lie continuously on the line segment L, occupying the interval $-a < x < a$ on the real axis, in an infinite plate, see Fig. 1. Denoting the intensity of dislocations of β_y, β_x and w on the line element ds at the point $P(x = s, y = 0)$ on L by $\phi_1(s)ds$, $\phi_3(s)ds$ and $\epsilon\phi_5(s)ds$ respectively, we have the following set of functions expressing the continuous arrays of dislocations [6]

$$\begin{aligned}
f'_c(z) &= -\frac{D}{2(\kappa+1)\pi}\int_{-a}^{a}\frac{\phi_1(s) - i\phi_3(s)}{s-z}ds,\\
h'_c(z) &= \frac{\kappa D}{2(\kappa+1)\pi}\int_{-a}^{a}\frac{\phi_1(s) + i\phi_3(s)}{s-z}ds + \frac{D}{2(\kappa+1)\pi}\int_{-a}^{a}\frac{s[\phi_1(s) - i\phi_3(s)]}{(s-z)^2}ds\\
&\quad - i\frac{\epsilon D}{2\pi}\int_{-a}^{a}\frac{\phi_5(s)}{(s-z)^2}ds,\\
\psi_c(z,\bar{z}) &= \frac{iD}{(\kappa+1)\pi}\int_{-a}^{a}[\phi_1(s) + i\phi_3(s)]\frac{1}{\epsilon}K_1\left(\frac{r_2}{\epsilon}\right)e^{i\theta_2}ds\\
&\quad - \frac{iD}{(\kappa+1)\pi}\int_{-a}^{a}[\phi_1(s) - i\phi_3(s)]\frac{1}{\epsilon}K_1\left(\frac{r_2}{\epsilon}\right)e^{-i\theta_2}ds - \frac{D}{(\kappa+1)\pi}\int_{-a}^{a}\phi_5(s)\frac{1}{\epsilon}K_0\left(\frac{r_2}{\epsilon}\right)ds,
\end{aligned} \tag{6}$$

where $r_2e^{i\theta_2} = z - s$. $K_n(z)$ are modified Bessel functions of the second kind. Substituting the above into eqn (4), we obtain the corresponding stress resultants as

$$M_x + M_y = \frac{2(\kappa-1)D}{(\kappa+1)^2\pi}\left\{\int_{-a}^{a}\frac{\phi_1(s) - i\phi_3(s)}{s-z}ds + \int_{-a}^{a}\frac{\phi_1(s) + i\phi_3(s)}{s-\bar{z}}ds\right\},$$

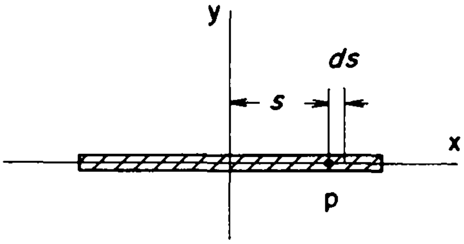

Fig. 1. Dislocation array in an elastic plate.

$$
\begin{aligned}
M_x - M_y - i2H_{xy} = \frac{2(\kappa-1)D}{(\kappa+1)^2\pi}\Bigg\{ &(z-\bar{z})\int_{-a}^{a} \frac{\phi_1(s)\mathrm{d}s}{(s-z)^2} + \frac{\kappa+1}{\kappa-1}\int_{-a}^{a} \phi_1(s) \\
&\times\left[\frac{1}{z-s} + \frac{\bar{z}-s}{(z-s)^2} - \frac{8\epsilon^2}{(z-s)^3} + \frac{1}{\epsilon}K_3\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i3\theta_2} - \frac{1}{\epsilon}K_1\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i\theta_2}\right]\mathrm{d}s \\
&- i2\int_{-a}^{a}\frac{\phi_3(s)\mathrm{d}s}{s-z} - i(z-\bar{z})\int_{-a}^{a}\frac{\phi_3(s)\mathrm{d}s}{(s-z)^2} \\
&+ i\frac{\kappa+1}{\kappa-1}\int_{-a}^{a}\phi_3(s)\mathrm{d}s\left[\frac{1}{z-s} - \frac{\bar{z}-s}{(z-s)^2} + \frac{8\epsilon^2}{(z-s)^3} - \frac{1}{\epsilon}K_3\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i3\theta_2} - \frac{1}{\epsilon}K_1\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i\theta_2}\right] \\
&+ i\frac{\kappa+1}{\kappa-1}\int_{-a}^{a}\phi_5(s)\left[\frac{2\epsilon}{(z-s)^2} - \frac{1}{\epsilon}K_2\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i2\theta_2}\right]\mathrm{d}s\Bigg\},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
V_x - iV_y = \frac{D}{(\kappa+1)\pi}\Bigg\{&\int_{-a}^{a}\phi_1(s)\left[\frac{2}{(z-s)^2} - \frac{1}{\epsilon^2}K_2\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i2\theta_2} + \frac{1}{\epsilon^2}K_0\left(\frac{r_2}{\epsilon}\right)\right]\mathrm{d}s + i\int_{a}^{a}\phi_3(s) \\
&\times\left[-\frac{2}{(z-s)^2} + \frac{1}{\epsilon^2}K_2\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i2\theta_2} + \frac{1}{\epsilon^2}K_0\left(\frac{r_2}{\epsilon}\right)\right]\mathrm{d}s - i\frac{1}{\epsilon}\int_{-a}^{a}\frac{\phi_5(s)\,\mathrm{d}s}{s-z} + i\int_{-a}^{a}\phi_5(s) \\
&\times\left[-\frac{1}{\epsilon(z-s)} + \frac{1}{\epsilon^2}K_1\left(\frac{r_2}{\epsilon}\right)\mathrm{e}^{-i\theta_2}\right]\mathrm{d}s\Bigg\}.
\end{aligned}
$$

The condition of single-valuedness of deflection and rotations due to the dislocation arrays on L can be expressed as [6]

$$
\int_{-a}^{a}\phi_j(s)\,\mathrm{d}s = 0\,(j = 1, 3), \int_{-a}^{a}[s\phi_3(s) + \epsilon\phi_5(s)]\,\mathrm{d}s = 0. \tag{8}
$$

4. PLAIN BENDING OF AN INFINITE PLATE CONTAINING TWO CRACKS

Let an unbounded plate of constant thickness h be weakened by two cracks L and L_1, the length of which is $2a$ and $2a_1$ respectively. The origin of Cartesian coordinates (x, y) is located at the center O of the crack L and the x axis coincides with the crack line. The center O_1 of the crack L_1 is given by the coordinates $z_0 = x_0 + iy_0 = re^{i\alpha}$ and the crack L_1 forms the angle β with the x axis, see Fig. 2. We also use local system of coordinates (x_1, y_1) in which the origin lies at O_1 and the x_1 axis runs along the crack line L_1. The relation between these two coordinates is given by

$$
z = z_1\,\mathrm{e}^{i\beta} + r\,\mathrm{e}^{i\alpha}. \tag{9}
$$

Provided that the load transmitted through the plate is the constant bending moment M_0 in the direction, making an angle γ with the x axis, and the rims of the cracks are free from traction, the boundary conditions of the problem are written as follows:

(a) At infinity $|z|\to\infty$,

$$
M_x = \frac{1}{2}M_0(1+\cos 2\gamma),\; M_y + iH_{xy} = \frac{1}{2}M_0(1-\mathrm{e}^{-i2\gamma}),\; V_x = V_y = 0. \tag{10}
$$

(b) On the rim of the crack L, $y = 0$, $|x| < a$,

$$
M_y + iH_{xy} = 0,\; V_y = 0. \tag{11}
$$

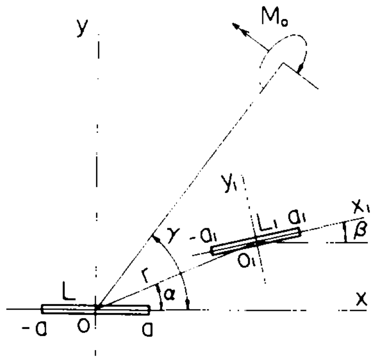

Fig. 2. Configuration and coordinate systems.

(c) On the rim of the crack L_1, $y_1 = 0$, $|x_1| < a_1$,

$$M_{y_1} + iH_{x_1y_1} = 0,\ V_{y_1} = 0. \tag{12}$$

In order to construct the set of three functions $f'(z)$, $h'(z)$ and $\psi(z, \bar{z})$ which satisfy all the boundary conditions (10), (11) and (12), we first consider the following functions

$$f_b'(z) = -\frac{\kappa+1}{\kappa-1}\frac{M_0}{8},\ h_b'(z) = -(\kappa+1)\frac{M_0}{8}e^{-i2\gamma},\ \psi_b(z, \bar{z}) = 0, \tag{13}$$

which express the state of plain bending of the crack-less plate and the corresponding stress resultants relative to the coordinates (x, y) are given by eqn (10). Hence, by eqn (5) we have the x_1-, y_1-components of stress resultants as

$$M_{x_1} = \frac{1}{2}M_0[1 + \cos 2(\gamma - \beta)],\ M_{y_1} + iH_{x_1y_1} = \frac{1}{2}M_0[1 - e^{-i2(\gamma-\beta)}],\ V_{x_1} = V_{y_1} = 0. \tag{14}$$

Thus the set of functions in eqn (13) does not satisfy the boundary conditions on the crack rims. For the accommodation of these conditions, we now consider the continuous arrays of dislocations on the crack lines L and L_1. If we denote the intensity of dislocations of β_y, β_x and w on the line element $\mathrm{d}s$ at the point $P(z = s)$ on L by $\phi_1(s)\,\mathrm{d}s$, $\phi_3(s)\,\mathrm{d}s$ and $\epsilon\phi_5(s)\,\mathrm{d}s$ respectively and the intensity of corresponding dislocations on the line element $\mathrm{d}s_1$ at the point $P_1(z_1 = s_1)$ on L_1 by $\phi_2(s_1)\mathrm{d}s_1$, $\phi_4(s_1)\mathrm{d}s_1$ and $\epsilon\phi_6(s_1)\mathrm{d}s_1$ respectively, the set of functions expressing the dislocation arrays on L is given by eqn (6) while the functions $f_{c_1}'(z_1)$, $h_{c_1}'(z_1)$ and $\psi_{c_1}(z_1, \bar{z}_1)$, which correspond to the dislocation arrays on the crack line L_1, are given by those in eqn (6), in which a, s, z, r_2, θ_2, ϕ_1, ϕ_3 and ϕ_5 are replaced by a_1, s_1, z_1, r_3, θ_3, ϕ_2, ϕ_4 and ϕ_6 respectively. Here r_3 and θ_3 are defined by

$$r_3 e^{i\theta_3} = z_1 - s_1. \tag{15}$$

As the stress resultants derived from these sets of functions tend to zero when $|z| \to \infty$, the superposition of these functions to those in eqn (13) does not disturb the boundary condition (10) at infinity.

The x-, y-components of the stress resultants corresponding to the set of functions (6) are given by eqn (7) and the x_1-, y_1-components are obtained by the transformation formulas (5). Similarly, the x_1-, y_1-components of the stress resultants due to the set of functions $f_{c_1}'(z_1)$, $h_{c_1}'(z_1)$ and $\psi_{c_1}(z_1, \bar{z}_1)$ can be written in the same form as those in eqn (7) and hence the x-, y-components are readily obtained by using the formulas (5).

By taking account of these and after some manipulations, from boundary conditions (11) and (12) we have

$$\frac{(\kappa+1)^2 M_0}{4(\kappa-1)D}(1 - e^{-i2\gamma}) + \frac{1}{\pi}\left\{\int_{-a}^{a} \frac{\phi_1(s) + i\phi_3(s)}{s - x}\,\mathrm{d}s + \int_{-a}^{a} f_{11}(x, s)\phi_1(s)\,\mathrm{d}s\right.$$

$$+i\int_{-a}^{a}[f_{33}(x,s)\phi_3(s)+f_{35}(x,s)\phi_5(s)]\,\mathrm{d}s$$

$$+\sum_{n=1}^{3}\int_{-a_1}^{a_1}[f_{1(2n)}(x,s_1)+if_{3(2n)}(x,s_1)]\phi_{2n}(s_1)\,\mathrm{d}s_1\Big\}=0,\,(|x|<a) \quad (16)$$

$$\frac{1}{\pi}\left\{\int_{-a}^{a}\frac{\phi_5(s)}{s-x}\mathrm{d}s+\frac{1}{\epsilon}\int_{-a}^{a}\phi_3(s)\ln|x-s|\,\mathrm{d}s+\int_{-a}^{a}[f_{53}(x,s)\phi_3(s)+f_{55}(x,s)\phi_5(s)]\,\mathrm{d}s\right.$$

$$\left.+\sum_{n=1}^{3}\int_{-a_1}^{a_1}f_{5(2n)}(x,s_1)\phi_{2n}(s_1)\,\mathrm{d}s_1\right\}=0,\,(|x|<a) \quad (17)$$

$$\frac{(\kappa+1)^2M_0}{4(\kappa-1)D}[1-\mathrm{e}^{i2(\gamma-\beta)}]+\frac{1}{\pi}\left\{\int_{-a_1}^{a_1}\frac{\phi_2(s_1)+i\phi_4(s_1)}{s_1-x_1}\mathrm{d}s_1+\int_{-a_1}^{a_1}f_{22}(x_1,s_1)\phi_2(s_1)\,\mathrm{d}s_1\right.$$

$$+i\int_{-a_1}^{a_1}[f_{44}(x_1,s_1)\phi_4(s_1)+f_{46}(x_1,s_1)\phi_6(s_1)]\,\mathrm{d}s_1$$

$$\left.+\sum_{n=1}^{3}\int_{-a}^{a}[f_{2(2n-1)}(x_1,s)+if_{4(2n-1)}(x_1,s)]\phi_{2n-1}(s)\,\mathrm{d}s\right\}=0,\,(|x_1|<a_1) \quad (18)$$

$$\frac{1}{\pi}\left\{\int_{-a_1}^{a_1}\frac{\phi_6(s_1)}{s_1-x_1}\mathrm{d}s_1+\frac{1}{\epsilon}\int_{-a_1}^{a_1}\phi_4(s_1)\ln|x_1-s_1|\,\mathrm{d}s_1+\int_{-a_1}^{a_1}[f_{64}(x_1,s_1)\phi_4(s_1)+f_{66}(x_1,s_1)\phi_6(s_1)]\,\mathrm{d}s_1\right.$$

$$\left.+\sum_{n=1}^{3}\int_{-a}^{a}f_{6(2n-1)}(x_1,s)\phi_{2n-1}(s)\,\mathrm{d}s\right\}=0,\,(|x_1|<a_1) \quad (19)$$

where the first integrals are understood to be the Cauchy principal value and

$$f_{11}(x,s)=\frac{\kappa+1}{\kappa-1}\frac{2}{x-s}\tilde{K}_2(\zeta),\quad f_{33}(x,s)=\frac{\kappa+1}{\kappa-1}\frac{2}{x-s}\left[\tilde{K}_2(\zeta)+\frac{\zeta}{2}\tilde{K}_1(\zeta)\right],$$

$$f_{35}(x,s)=\frac{\kappa+1}{\kappa-1}\frac{1}{2\epsilon}\left[\tilde{K}_2(\zeta)-\frac{1}{2}\right], \quad (20)$$

$$f_{53}(x,s)=-[\tilde{K}_2(\zeta)+\tilde{K}_0(\zeta)]/\epsilon,\quad f_{55}(x,s)=-\operatorname{sgn}(x-s)\tilde{K}_1(\zeta)/\epsilon,$$

$$f_{12}(x,s_1)+if_{32}(x,s_1)=-\frac{1}{2\rho_3}[\mathrm{e}^{i\chi_3}+\mathrm{e}^{-i\chi_3}+\mathrm{e}^{-i(\chi_3+2\beta)}-\mathrm{e}^{-i(3\chi_3+2\beta)}]$$

$$-\frac{\kappa+1}{2(\kappa-1)}\left[\frac{1}{\rho_3}(\mathrm{e}^{-i\chi_3}+\mathrm{e}^{-i3\chi_3})-\frac{8\epsilon^2}{\rho_3^3}\mathrm{e}^{i3\chi_3}\right.$$

$$\left.+\frac{1}{\epsilon}K_3\left(\frac{\rho_3}{\epsilon}\right)\mathrm{e}^{i3\chi_3}-\frac{1}{\epsilon}K_1\left(\frac{\rho_3}{\epsilon}\right)\mathrm{e}^{-i\chi_3}\right]\mathrm{e}^{-i2\beta},$$

$$f_{14}(x,s_1)+if_{34}(x,s_1)=-\frac{i}{2\rho_3}[\mathrm{e}^{i\chi_3}-\mathrm{e}^{-i\chi_3}+\mathrm{e}^{-i(\chi_3+2\beta)}+\mathrm{e}^{-i(3\chi_3+2\beta)}] \quad (21)$$

$$-\frac{i(\kappa+1)}{2(\kappa-1)}\left[\frac{1}{\rho_3}(\mathrm{e}^{-i\chi_3}-\mathrm{e}^{-i3\chi_3})+\frac{8\epsilon^2}{\rho_3^3}\mathrm{e}^{-i3\chi_3}\right.$$

$$\left.-\frac{1}{\epsilon}K_3\left(\frac{\rho_3}{\epsilon}\right)\mathrm{e}^{-i3\chi_3}-\frac{1}{\epsilon}K_1\left(\frac{\rho_3}{\epsilon}\right)\mathrm{e}^{-i\chi_3}\right]\mathrm{e}^{-i2\beta},$$

$$f_{16}(x,s_1)+if_{36}(x,s_1)=\frac{i(\kappa+1)}{2(\kappa-1)}\mathrm{e}^{-i2(\chi_3+\beta)}\left[-\frac{2\epsilon}{\rho_3^2}+\frac{1}{\epsilon}K_2\left(\frac{\rho_3}{\epsilon}\right)\right],$$

$$f_{52}(x,s_1)+if_{54}(x,s_1)=i\,\mathrm{e}^{-i(2\chi_3+\beta)}\left[\frac{2\epsilon}{\rho_3^2}-\frac{1}{\epsilon}K_2\left(\frac{\rho_3}{\epsilon}\right)\right]-i\frac{1}{\epsilon}K_0\left(\frac{\rho_3}{\epsilon}\right)\mathrm{e}^{i\beta},$$

$$f_{56}(x,s_1)=-\frac{1}{\epsilon}K_1\left(\frac{\rho_3}{\epsilon}\right)\cos(\chi_3+\beta).$$

Here the following notations are used

$$\zeta = |x - s|/\epsilon,\ \tilde{K}_0(\zeta) = K_0(\zeta) + \ln(\zeta/2) + \gamma_0,$$
$$\tilde{K}_1(\zeta) = K_1(\zeta) - (1/\zeta),\ \tilde{K}_2(\zeta) = K_2(\zeta) - (2/\zeta^2) + (1/2), \tag{22}$$
$$\rho_3\,e^{i\chi_3} = e^{-i\beta}(x - r\,e^{i\alpha} - s_1\,e^{i\beta}), \tag{23}$$

where γ_0 is Euler's constant. The expressions for $f_{22}(x_1, s_1)$, $f_{44}(x_1, s_1)$, $f_{46}(x_1, s_1)$, $f_{64}(x_1, s_1)$ and $f_{66}(x_1, s_1)$ are given by those for $f_{11}, f_{33}, f_{35}, f_{53}$ and f_{55} in eqn (20) respectively, in which x and s are replaced by x_1 and s_1. Furthermore, the expressions for $f_{21} + if_{41}, f_{23} + if_{43}, f_{25} + if_{45}, f_{61} + if_{63}$ and f_{65} are given by those for $f_{12} + if_{32}, f_{14} + if_{34}, f_{16} + if_{36}, f_{52} + if_{54}$ and f_{56} in eqn (21) respectively, in which x, s_1, ρ_3, χ_3 and β are replaced by x_1, s, ρ_2, χ_2 and $-\beta$ respectively, ρ_2 and χ_2 being defined by

$$\rho_2\,e^{i\chi_2} = e^{i\beta}(x_1 + r\,e^{i(\alpha-\beta)} - s\,e^{-i\beta}). \tag{24}$$

Now the following substitutions may be made:

$$s/a = s_1/a_1 = S,\ x/a = x_1/a_1 = X,\ a_1/a = L,$$
$$\epsilon/a = c,\ r/a = R,\ \rho_2/a = R_2,\ \rho_3/a = R_3, \tag{25}$$

$$\begin{pmatrix}\phi_{2n-1}(s)\\ \phi_{2n}(s_1)\end{pmatrix} = -\frac{(\kappa+1)^2 M_0}{2(\kappa-1)D}\begin{pmatrix}\Phi_{2n-1}(S)\\ \Phi_{2n}(S)\end{pmatrix},\ (n = 1,2,3) \tag{26}$$

$$\begin{aligned}
&(f_{11}, f_{21}, f_{41}, f_{61}; f_{23}, f_{33}, f_{43}, f_{53}, f_{63}; f_{25}, f_{35}, f_{45}, f_{55}, f_{65})\\
&\quad = (F_{11}, F_{21}, F_{41}, F_{61}; F_{23}, F_{33}, F_{43}, F_{53}, F_{63}; F_{25}, F_{35}, F_{45}, F_{55}, F_{56})/a,\\
&(f_{12}, f_{22}, f_{32}, f_{52}; f_{14}, f_{34}, f_{44}, f_{54}, f_{64}; f_{16}, f_{36}, f_{46}, f_{56}, f_{66})\\
&\quad = (F_{12}, F_{22}, F_{32}, F_{52}; F_{14}, F_{34}, F_{44}, F_{54}, F_{64}; F_{16}, F_{36}, F_{46}, F_{56}, F_{66})/a_1,\\
&h_{ij} = (\delta_{5i}\delta_{3j}/c) + (\delta_{6i}\delta_{4j}L/c),\ (i, j = 1, 2, 3, \ldots, 6)\\
&\left(\frac{1}{2}(1 - \cos 2\gamma), \frac{1}{2}[1 - \cos 2(\gamma - \beta)], \frac{1}{2}\sin 2\gamma, \frac{1}{2}\sin 2(\gamma - \beta), 0, 0\right) = [C_1, C_2, C_3, C_4, C_5, C_6],
\end{aligned} \tag{27}$$

where δ_{ij} is the Kronecker delta. Moreover, $F_{ij}(X, S)$ $(i, j = 1, 2, 3, \ldots, 6)$ which are not defined in eqn (27) are to be zero. By these substitutions, eqns (16) to (19) are changed into

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\Phi_i(S)\,dS}{S - X} + \frac{1}{\pi}\sum_{j=1}^{6}\left[\int_{-1}^{1} F_{ij}(X, S)\Phi_j(S)\,dS + h_{ij}\int_{-1}^{1}\Phi_j(S)\ln|X - S|\,dS\right] = C_i,\ (|X| < 1,\ i = 1, 2, 3, \ldots, 6) \tag{28}$$

where the Cauchy principal value is taken for the first integral.

By observing the behavior of modified Bessel function of the second kind for small argument, the functions $F_{ij}(X, S)$ are proved to be bounded in the closed interval $-1 \leqq X, S \leqq 1$ and thus the six equations in eqn (28) are a system of singular integral equations with Cauchy type kernels.

By the same substitution, the conditions of single-valuedness of deflection and rotations due to the dislocation arrays on L and L_1, i.e. eqn (8) and its equivalents, are transformed into

$$\int_{-1}^{1}\Phi_j(S)\,dS = 0,\ (j = 1, 2, 3, 4)\ \int_{-1}^{1}[S\Phi_3(S) + c\Phi_5(S)]\,dS = 0,\ \int_{-1}^{1}[S\Phi_4(S) + (c/L)\Phi_6(S)]\,dS = 0. \tag{29}$$

These are the additional conditions which fix the unknown functions $\Phi_j(S)$ $(j = 1, 2, 3, \ldots, 6)$. Thus the problem is reduced to the solution of the set of singular integral eqns (28) under the additional conditions (29), which has already been discussed by Erdogan[8].

Following his technique, we assume the unknown dislocation densities in the form

$$\Phi_j(S) = \sum_{n=0}^{\infty} a_{jn} T_n(S)/\sqrt{(1-S^2)}, \quad (j = 1, 2, 3, \ldots, 6) \tag{30}$$

where a_{jn} are unknown constants and $T_n(S)$ are Chebyshev polynomials of the first kind. Substituting the above into eqn (29) and integrating, we have

$$a_{j0} = 0 (j = 1, 2, 3, 4), a_{31} + 2ca_{50} = 0, a_{41} + 2(c/L)a_{60} = 0. \tag{31}$$

Substituting eqn (30) into eqn (28) and considering the orthogonality relations of Chebyshev polynomials[9] and eqn (31), we finally get the following set of linear equations for the unknown constants

$$\sum_{j=1}^{6} \sum_{n=1}^{\infty} [\delta_{ij}\delta_{kn} + d_{ijkn}] b_{jn} = C_i \delta_{1k}, \quad (i = 1, 2, \ldots, 6; k = 1, 2, \ldots\ldots) \tag{32}$$

where

$$b_{jn} = a_{jn} - (\delta_{5j} + L\delta_{6j}) \left[\frac{1}{2(n-1)c} a_{(j-2)(n-1)}(1 - \delta_{1n}) - \frac{1}{2(n+1)c} a_{(j-2)(n+1)} \right], \tag{33}$$

and

$$d_{ijkn} = c_{ijkn} - (\delta_{3j} + L\delta_{4j})[c_{i(j+2)k(n-1)} - c_{i(j+2)k(n+1)}] \frac{1}{2nc}. \tag{34}$$

The coefficients c_{ijkn} are given by

$$c_{ijkn} = \frac{2}{\pi^2} \int_{-1}^{1} U_{k-1}(X)\sqrt{(1-X^2)} \left[\int_{-1}^{1} \frac{T_n(S)}{\sqrt{(1-S^2)}} F_{ij}(X, S)\, dS \right] dX, \tag{35}$$

where $U_n(X)$ are Chebyshev polynomials of the second kind. The integrals in eqn (35) are of Gauss–Chebyshev type and may easily be evaluated by employing proper quadrature formulas[10].

5. BEHAVIOR OF STRESS RESULTANTS NEAR CRACK TIPS

We will proceed to the study of the asymptotic behavior of stress resultants in the vicinity of the vertices of cracks.

It is evident that the set of functions $f_c'(z)$, $h_c'(z)$ and $\psi_c(z, \bar{z})$ in eqn (6) takes the leading role in the singular character of the stress resultants near the crack L, while in the neighborhood of $L_1 f_{c_1}'(z_1)$, $h_{c_1}'(z_1)$ and $\psi_{c_1}(z_1, \bar{z}_1)$ make an important contribution. Consequently, by the same procedure as that in the case of a single crack in an infinite plate[6], we can easily obtain the asymptotic behavior of stress resultants near the crack tips. Especially the moment intensity factor of opening mode k_{m1}, of sliding mode k_{m2} and the shearing force intensity factor k_ν at the ends $A(z = a)$ and $B(z = -a)$ of the crack L are given by

$$\begin{pmatrix} k_{m1}(A) \\ k_{m1}(B) \end{pmatrix} = M_0\sqrt{(a)} \sum_{n=1}^{\infty} (\pm 1)^{n+1} b_{1n}, \begin{pmatrix} k_{m2}(A) \\ k_{m2}(B) \end{pmatrix} = M_0\sqrt{(a)} \sum_{n=1}^{\infty} (\pm 1)^{n+1} b_{3n},$$
$$\begin{pmatrix} k_\nu(A) \\ k_\nu(B) \end{pmatrix} = -\frac{M_0}{\epsilon}\sqrt{(a)} \frac{\kappa + 1}{2(\kappa - 1)} \sum_{n=1}^{\infty} (\pm 1)^{n+1} b_{5n}. \tag{36}$$

The corresponding quantities at the ends $A_1(z_1 = a_1)$ and $B_1(z_1 = -a_1)$ of the crack L_1 can be obtained by the above expressions in which A, B, a, b_{1n}, b_{3n} and b_{5n} are replaced by A_1, B_1, a_1, b_{2n}, b_{4n} and b_{6n} respectively.

6. NUMERICAL RESULTS

Following the aforementioned analysis, some numerical calculations are performed for two fundamental cases of crack geometry, i.e.

(a) a pair of equal colinear cracks,
(b) two equal parallel cracks without stagger.

6.1 *The case of a pair of colinear cracks*

In this particular case, we have $\alpha = 0$ in eqn (9). Provided $\beta = \pi$ in the same equation, from eqns (23), (24) and (25) we get

$$R_1 = R - X - LS,\ R_2 = R - LX - S,\ \chi_1 = \chi_2 = 0. \tag{37}$$

Hence in eqn (27) we have

$$C_1 = C_2 = \frac{1}{2}(1 - \cos 2\gamma),\ C_3 = C_4 = \frac{1}{2}\sin 2\gamma,\ C_5 = C_6 = 0. \tag{38}$$

For the elements of the coefficient matrix $[d_{ijkn}]$ in eqn (34) we obtain

$$d_{14kn} = d_{16kn} = d_{23kn} = d_{25kn} = d_{32kn} = d_{41kn} = d_{52kn} = d_{61kn} = 0. \tag{39}$$

Moreover the following elements are originally zero

$$d_{13kn} = d_{15kn} = d_{24kn} = d_{26kn} = d_{31kn} = d_{42kn} = d_{51kn} = d_{62kn} = 0. \tag{40}$$

Therefore the system of linear eqns (32) are divided into two parts as

$$\sum_{j=1}^{2}\sum_{n=1}^{\infty}(\delta_{ij}\delta_{kn} + d_{ijkn})B_{jn} = \delta_{1k},\ (i = 1, 2;\ k = 1, 2, \ldots\ldots) \tag{41}$$

$$\sum_{j=3}^{6}\sum_{n=1}^{\infty}(\delta_{ij}\delta_{kn} + d_{ijkn})B_{jn} = \delta_{1k}(\delta_{3i} + \delta_{4i}),\ (i = 3, 4, 5, 6;\ k = 1, 2, \ldots) \tag{42}$$

where

$$B_{jn} = b_{jn}\Big/\frac{1}{2}(1 - \cos 2\gamma),\ (j = 1, 2),\ B_{jn} = b_{jn}\Big/\frac{1}{2}\sin 2\gamma,\ (j = 3, 4, 5, 6). \tag{43}$$

In this case the moment- and shearing-force-intensity factors in eqn (36) are rewritten as

$$\begin{aligned}
\begin{pmatrix} k_{m1}(A) \\ k_{m1}(B) \end{pmatrix} &= \frac{1}{2}M_0\sqrt{(a)}(1 - \cos 2\gamma)\begin{pmatrix} F_{1A} \\ F_{1B} \end{pmatrix},\ \begin{pmatrix} F_{1A} \\ F_{1B} \end{pmatrix} = \sum_{n=1}^{\infty}(\pm 1)^{n-1}B_{1n},\\
\begin{pmatrix} k_{m2}(A) \\ k_{m2}(B) \end{pmatrix} &= \frac{1}{2}M_0\sqrt{(a)}\sin 2\gamma\begin{pmatrix} F_{2A} \\ F_{2B} \end{pmatrix},\ \begin{pmatrix} F_{2A} \\ F_{2B} \end{pmatrix} = \sum_{n=1}^{\infty}(\pm 1)^{n-1}B_{3n},\\
\begin{pmatrix} k_{\nu}(A) \\ k_{\nu}(B) \end{pmatrix} &= \frac{1}{2}\frac{M_0}{\epsilon}\sqrt{(a)}\sin 2\gamma\begin{pmatrix} F_{3A} \\ F_{3B} \end{pmatrix},\ \begin{pmatrix} F_{3A} \\ F_{3B} \end{pmatrix} = -\frac{\kappa + 1}{2(\kappa - 1)}\sum_{n=1}^{\infty}(\pm 1)^{n+1}B_{5n}.
\end{aligned} \tag{44}$$

The equivalents for the ends A_1 and B_1 of the crack L_1 are given by the above equation in which A, B, a, B_{1n}, B_{3n} and B_{5n} are changed into A_1, B_1, a_1, B_{2n}, B_{4n} and B_{6n} respectively. It is evident that in this crack geometry the effect of the load direction γ on the intensity factors is separated from other factors such as configuration parameters a/r, ϵ/a, a_1/a and reduced Poisson's ratio κ. If the plain bending takes place along the crack line, i.e. $\gamma = 0$, we have $k_{m1} = k_{m2} = k_{\nu} = 0$ and there exists no singularity. If the constant bending moment M_0 is transmitted in the direction perpendicular to the crack line ($\gamma = \pi/2$), we get $k_{m2} = k_{\nu} = 0$.

If, further, two cracks are equal in length, i.e. $a = a_1$, we obtain

$$B_{1n} = B_{2n},\quad B_{3n} = B_{4n},\quad B_{5n} = B_{6n}. \tag{45}$$

and the set of linear equations for these unknown coefficients is simplified as

$$
\begin{aligned}
&\sum_{n=1}^{\infty} (\delta_{kn} + d_{11kn} + d_{12kn}) B_{1n} = \delta_{1k}, \\
&\sum_{n=1}^{\infty} [(\delta_{kn} + d_{33kn} + d_{34kn}) B_{3n} + (d_{35kn} + d_{36kn}) B_{5n}] = \delta_{1k}, \quad (k = 1, 2, ..) \\
&\sum_{n=1}^{\infty} [(d_{53kn} + d_{54kn}) B_{3n} + (\delta_{kn} + d_{55kn} + d_{56kn}) B_{5n}] = 0.
\end{aligned} \tag{46}
$$

In order to study the effect of the neighboring cracks, some numerical works are carried out for the present case. Once the geometrical parameters ϵ/a, a/r and Poisson's ratio ν are specified, the set of linear eqns (46) is solved by an approximate method in which only the first N equations containing only the first N unknowns are taken. The computation reveals that the value of N needed to achieve a particular level of accuracy is strongly dependent on ϵ/a and this dependence is quite similar as that in the case of $a/r \to \infty$ [6]. An example is shown in Table 1, which gives the values of F_{1A} against N for three values of ϵ/a, a/r and ν being 0·475 and 0 respectively.

Table 1. Effect of N on moment intensity factor F_{1A} ($\nu = 0$, $a/r = 0{\cdot}475$)

N	ϵ/a		
	1.0	0.3	0.1
20	1.3667	1.1874	1.2099
40	1.3667	1.1875	1.2108
60	—	1.1876	1.2111

Figures 3 and 4 show the numerical results of F_1 and F_2 against a/r for various values of ϵ/a and ν. In these figures, the mark on the line of $a/r = 0{\cdot}5$ means the reduced value of corresponding intensity factor ratios for a single crack of length $4a$ in an unbounded plate of thickness h [6]. The values of F_1 and F_2 are plotted versus ϵ/a in Fig. 5. Since the shearing force intensity factor ratio F_3 remains in the range of comparatively small values, the variation of F_3 due to a/r and ϵ/a is not shown here for brevity.

Inspection of these figures will reveal the following facts:

(a) With the approach of two cracks, the moment intensity factor ratios F_1 and F_2 at the inside tip A increase markedly.

(b) When a/r is not so large, F_{1A} decreases monotonically with the decrease of ϵ/a, while in the case of $a/r = 0{\cdot}475$ F_{1A} attains its maximum at a certain value of ϵ/a. It may be understood that the effect of the adjacent crack on the moment intensity factor k_1 at the inner crack tip is much larger when the thickness ratio ϵ/a becomes comparatively small.

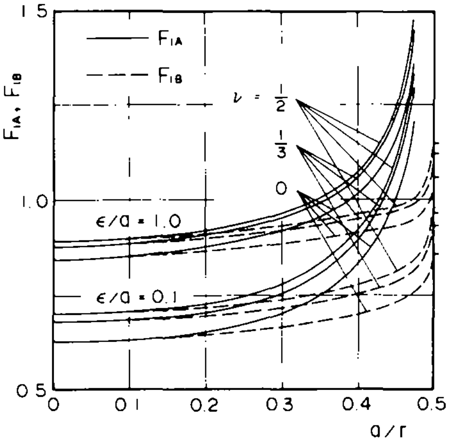

Fig. 3. Moment intensity factor ratio F_1 vs crack length ratio a/r.

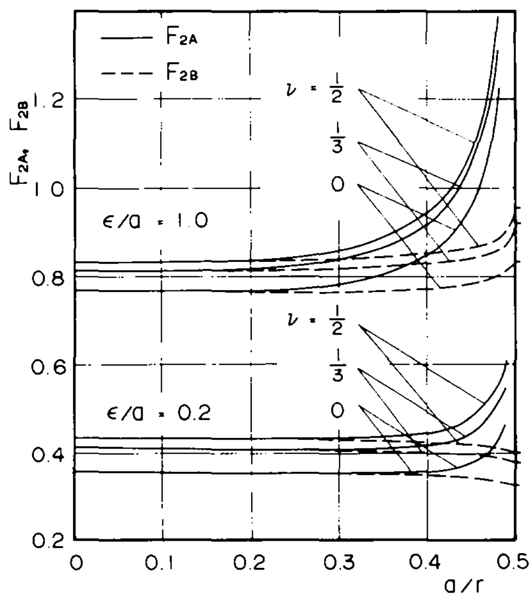

Fig. 4. Moment intensity factor ratio F_2 vs crack length ratio a/r.

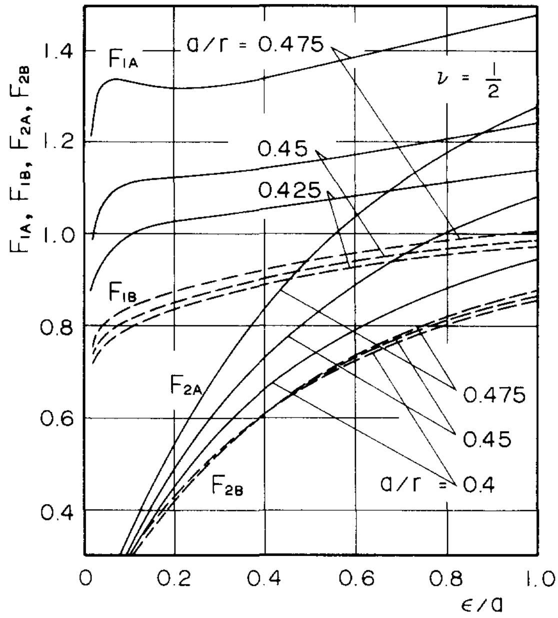

Fig. 5. Moment intensity factor ratios F_1 and F_2 vs plate thickness ratio ϵ/a.

6.2. *The case of two parallel cracks without stagger*

In this case we have $\alpha = \pi/2$ and it is convenient to put $\beta = \pi$. As expected by the symmetry of the problem, it is easily proved that some of the elements of coefficient matrix $[d_{ijkn}]$ vanish and by putting

$$[b_{i(2n-1)}, b_{j(2n)}] = \frac{1}{2}(1-\cos 2\gamma)\,[B_{i(2n-1)}, B_{j(2n)}], \qquad \begin{pmatrix} i = 1,2,5,6 \\ j = 3,4 \end{pmatrix} \qquad (47)$$
$$[b_{i(2n)}, b_{j(2n-1)}] = \frac{1}{2}\sin 2\gamma\,[B_{i(2n)}, B_{j(2n-1)}],$$

the set of linear equations can be divided into two groups. Further, the intensity factors at the crack tips A and B can be expressed as

$$\begin{pmatrix} k_{m1}(A) \\ k_{m1}(B) \end{pmatrix} = \frac{1}{2} M_0\sqrt{(a)}\,[(1-\cos 2\gamma)F^*_{11} \pm \sin 2\gamma F^*_{12}],$$
$$\begin{pmatrix} k_{m2}(A) \\ k_{m2}(B) \end{pmatrix} = \frac{1}{2} M_0\sqrt{(a)}\,[\pm(1-\cos 2\gamma)F^*_{21} + \sin 2\gamma F^*_{22}], \qquad (48)$$
$$\begin{pmatrix} k_{\nu}(A) \\ k_{\nu}(B) \end{pmatrix} = \frac{1}{2}\frac{M_0}{\epsilon}\sqrt{(a)}\,[(1-\cos 2\gamma)F^*_{31} \pm \sin 2\gamma F^*_{32}],$$

where

$$F^*_{11} = \sum_{n=1}^{\infty} B_{1(2n-1)},\quad F^*_{12} = \sum_{n=1}^{\infty} B_{1(2n)},\quad F^*_{21} = \sum_{n=1}^{\infty} B_{3(2n)},\quad F^*_{22} = \sum_{n=1}^{\infty} B_{3(2n-1)},$$
$$F^*_{31} = -\frac{\kappa+1}{2(\kappa-1)}\sum_{n=1}^{\infty} B_{5(2n-1)},\quad F^*_{32} = -\frac{\kappa+1}{2(\kappa-1)}\sum_{n=1}^{\infty} B_{5(2n)}. \qquad (49)$$

The corresponding quantities for the ends A_1 and B_1 of the other crack L_1 are given by the same replacement as before. Thus the influence of the loading direction γ on the intensity factors is separated from other factors, so that if $\gamma = \pi/2$, $k_{m1}(A)$, $k_{m2}(A)$ and $k_{\nu}(A)$ are controlled only by F^*_{11}, F^*_{21} and F^*_{31} respectively.

In the more particular case of equal crack length, there also exist the relations in eqn (45) and the sets of linear equations are simplified considerably. The first set which determines $B_{1(2n-1)}$, $B_{3(2n)}$ and $B_{5(2n-1)}$ can be written as

$$\begin{aligned}
&\sum_{n=1}^{\infty} [\{\delta_{kn} + d_{11(2k-1)(2n-1)} + d_{12(2k-1)(2n-1)}\}B_{1(2n-1)} \\
&\qquad + d_{14(2k-1)(2n)}B_{3(2n)} + d_{16(2k-1)(2n-1)}B_{5(2n-1)}] = \delta_{1k}, \\
&\sum_{n=1}^{\infty} [d_{32(2k)(2n-1)}B_{1(2n-1)} + \{\delta_{kn} + d_{33(2k)(2n)} + d_{34(2k)(2n)}\}B_{3(2n)} \qquad (k = 1, 2, \ldots) \\
&\qquad + \{d_{35(2k)(2n-1)} + d_{36(2k)(2n-1)}\}B_{5(2n-1)}] = 0, \\
&\sum_{n=1}^{\infty} [d_{52(2k-1)(2n-1)}B_{1(2n-1)} + \{d_{53(2k-1)(2n)} + d_{54(2k-1)(2n)}\}B_{3(2n)} \\
&\qquad + \{\delta_{kn} + d_{55(2k-1)(2n-1)} + d_{56(2k-1)(2n-1)}\}B_{5(2n-1)}] = 0.
\end{aligned} \tag{50}$$

The remaining set of linear equations for $B_{1(2n)}$, $B_{3(2n-1)}$ and $B_{5(2n)}$ is given by the above equations in which $(2k-1)$, $(2n-1)$, $2k$ and $2n$ are replaced by $2k$, $2n$, $(2k-1)$ and $(2n-1)$ respectively and the constant terms on the right hand side $(\delta_{1k}, 0, 0)$ are changed into $(0, \delta_{1k}, 0)$. These systems of linear equations may be solved by the approximate method mentioned previously.

Numerical results of F^*_{11}, F^*_{12} and F^*_{22} are shown versus the crack length-space ratio a/r for various values of ϵ/a and ν in Figs. 6 and 7, while they are plotted against the plate thickness ratio ϵ/a in Fig. 8. The values of F^*_{21}, F^*_{31} and F^*_{32} remain so small that the graphs for them are not shown here.

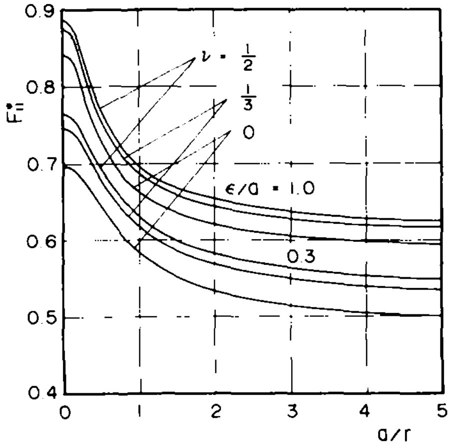

Fig. 6. Moment intensity factor ratio F^*_{11} vs crack length-space ratio a/r.

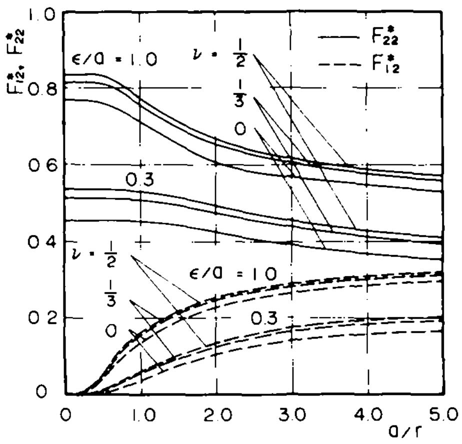

Fig. 7. Moment intensity factor ratios F^*_{12} and F^*_{22} vs crack length-space ratio a/r.

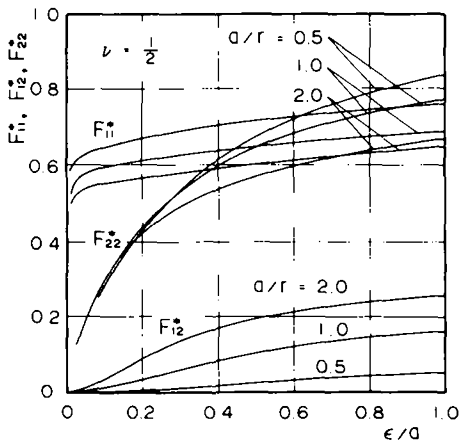

Fig. 8. Moment intensity factor ratios F^*_{11}, F^*_{12} and F^*_{22} vs plate thickness ratio ϵ/a.

Examination of these figures will show the following:

(a) The stress relieving effect of the neighboring parallel crack is recognized in the variation of F^*_{11} and F^*_{22}, and this effect for F^*_{11} is remarkable in the range $a/r < 1$.

(b) The values of F^*_{11} and F^*_{22} decrease monotonically with the decrease of ϵ/a.

7. UNIFORM TWISTING PROBLEM

As to the case where the constant twisting moment H_0 is transmitted through an unbounded plate, weakened by two arbitrarily situated cracks L and L_1, in the direction forming an angle γ with the crack L, the analysis can be performed quite parallel with the previous case. Instead of the set of three functions in eqn (13), we shall start with the following functions

$$f_t'(z) = 0, \quad h_t'(z) = i(\kappa + 1)H_0\,\mathrm{e}^{-i2\gamma}/4, \quad \psi_t(z, \bar{z}) = 0. \tag{51}$$

In order to satisfy the traction free conditions on the crack rims in eqns (11) and (12), we may as well superimpose the continuous arrays of dislocations on two crack lines. Rewriting the unknown dislocation density as

$$\begin{pmatrix} \phi_{2n-1}(s) \\ \phi_{2n}(s_1) \end{pmatrix} = -\frac{(\kappa + 1)^2 H_0}{2(\kappa - 1)D} \begin{pmatrix} \Phi_{2n-1}(S) \\ \Phi_{2n}(S) \end{pmatrix}, \quad (n = 1, 2, 3) \tag{52}$$

and assuming the expansion in eqn (30), we arrive at the set of linear eqns (32) for the unknown coefficients b_{jn}, in which constant terms in the right hand side are replaced by

$$[C_1, C_2, C_3, C_4, C_5, C_6] = [\sin 2\gamma, \sin 2(\gamma - \beta), \cos 2\gamma, \cos 2(\gamma - \beta), 0, 0] \tag{53}$$

By the use of coefficients b_{jn}, thus determined, the intensity factors at the crack tips can be obtained by eqn (36) and their equivalents, in which M_0 is replaced by H_0. Consequently in the particular cases of crack geometry, considered before, the intensity factors are given by eqns (44) and (48) respectively in which $[\frac{1}{2}M_0(1 - \cos 2\gamma), \frac{1}{2}M_0 \sin 2\gamma]$ are replaced by $[H_0 \sin 2\gamma, H_0 \cos 2\gamma]$.

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